Closed-Form Matting

CVFX @ NTHU

30 April 2015

The Paper

- *A Closed Form Solution to Natural Image Matting*
  - Levin, Lischinski, and Weiss
  - CVPR 2006

Compositing Equation

$$I_i = \alpha_i F_i + (1 - \alpha_i) B_i$$

composite (I_i)

alpha channel (α_i)

foreground (F_i)

background (B_i)

natural matting:

α_i, F_i, and B_i are unknowns

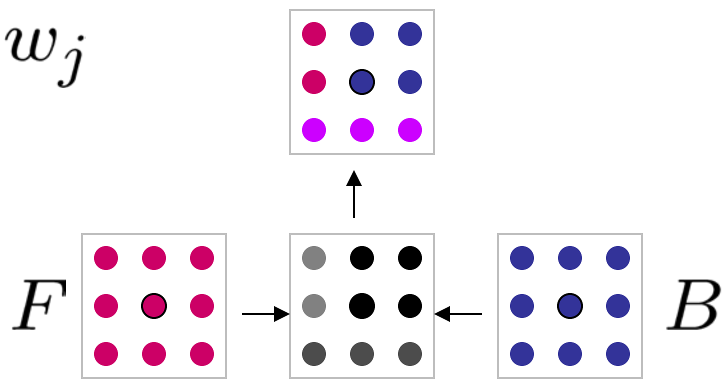
for a 3 channel color image
at each pixel there are 3 equations and 7 unknowns

Alpha Matting of Grayscale Images

- Assumption: both F and B are approximately constant over a small window around each pixel
  - Locally smooth → linear relation

$$\alpha_i \approx aI_i + b, \quad \forall i \in w$$ ← a small window

$$a = \frac{1}{F-B} \qquad b = -\frac{B}{F-B}$$



$$I_i \approx \alpha_i F + (1-\alpha_i)B$$
$$I_i \approx \alpha_i(F-B) + B$$
$$\alpha_i \approx \frac{1}{F-B} I_i - \frac{B}{F-B}$$

Optimization

- Minimize the cost function

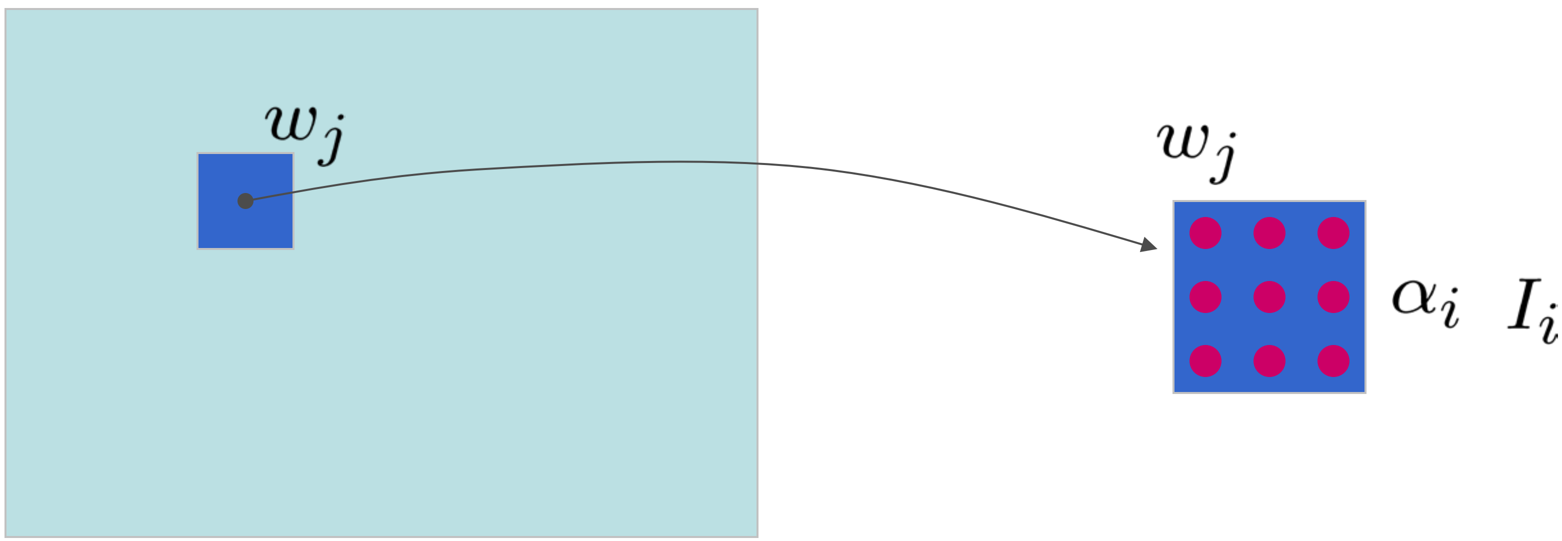
$$J(\alpha, a, b) = \sum_{j \in I} \left(\sum_{i \in w_j} (\alpha_i - a_j I_i - b_j)^2 + \epsilon a_j^2 \right)$$

where w_j is a small window around pixel j

a regularization term on a:

minimizing the norm of a biases the solution towards smoother α mattes $\alpha_i \approx aI_i + b, \quad \forall i \in w$

$a \ll 0$ implies that F and B are very different



$$J(\alpha, a, b) = \sum_{j \in I} \left(\sum_{i \in w_j} (\alpha_i - a_j I_i - b_j)^2 + \epsilon a_j^2 \right)$$

$$J\left(\alpha, a, b\right) = \sum_{j \in I} \left(\sum_{i \in w_j} \left(\alpha_i - a_j I_i - b_j\right)^2 + \epsilon a_j^2 \right)$$

$$J(\alpha, a, b) = \sum_k \left\| \begin{pmatrix} I_1 & 1 \\ \vdots & \vdots \\ I_{|w_k|} & 1 \\ \sqrt{\epsilon} & 0 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} - \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{|w_k|} \\ 0 \end{pmatrix} \right\|^2$$

$$J\left(\alpha, a, b\right) = \sum_k \left\| G_k \begin{bmatrix} a_k \\ b_k \end{bmatrix} - \bar{\alpha}_k \right\|^2$$

- For a given alpha matte the optimal pair a_k^*, b_k^* inside each window w_k is the solution to the least squares problem

$$
\begin{aligned}
(a_k^*, b_k^*) &= \operatorname{argmin} \left\| G_k \begin{bmatrix} a_k \\ b_k \end{bmatrix} - \bar{\alpha}_k \right\|^2 \\
&= (G_k^T G_k)^{-1} G_k^T \bar{\alpha}_k
\end{aligned}
$$

Substituting a_k^*, b_k^*

$$\begin{pmatrix} a_k^* \\ b_k^* \end{pmatrix} = (G_k^T G_k)^{-1} G_k^T \bar{\alpha}_k$$

$$J(\alpha, a^*, b^*) = \sum_k \left\| G_k \begin{pmatrix} a_k^* \\ b_k^* \end{pmatrix} - \bar{\alpha}_k \right\|^2$$

$$J(\alpha) = \sum_k \left\| \left(G_k (G_k^T G_k)^{-1} G_k^T - \mathbf{I} \right) \bar{\alpha}_k \right\|^2$$

$$= \sum_k \bar{\alpha}_k^T \bar{G}_k^T \bar{G}_k \bar{\alpha}_k$$

$$\bar{G}_k = \mathbf{I} - G_k (G_k^T G_k)^{-1} G_k^T$$

$$
\begin{aligned}
\bar{G}_k^T \bar{G}_k &= (\mathbf{I} - G_k(G_k^T G_k)^{-1} G_k^T)^T (\mathbf{I} - G_k(G_k^T G_k)^{-1} G_k^T) \\
&= \mathbf{I} - G_k(G_k^T G_k)^{-1} G_k^T
\end{aligned}
$$

$$
G_k = \begin{pmatrix} I_1 & 1 \\ \vdots & \vdots \\ I_{|w_k|} & 1 \\ \sqrt{\epsilon} & 0 \end{pmatrix}
$$

the (i,j)-th element of $\mathbf{I} - G_k(G_k^T G_k)^{-1} G_k^T$ is

$$
\delta_{ij} - \frac{1}{|w_k|}\left(1 + \frac{1}{\frac{\epsilon}{|w_k|} + \sigma_k^2}(I_i - \mu_k)(I_j - \mu_k)\right)
$$

The (i,j) element

$$\left(\mathbf{I} - G_k\left(G_k^T G_k\right)^{-1} G_k^T\right)_{ij}$$

$$= \delta_{ij} - (I_i \ 1)\begin{pmatrix} \sum_n^{|w_k|} I_n^2 + \epsilon & \sum_n^{|w_k|} I_n \\ \sum_n^{|w_k|} I_n & |w_k| \end{pmatrix}^{-1}\begin{pmatrix} I_j \\ 1 \end{pmatrix}$$

$$\mathbf{I} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} I_1 & 1 \\ I_2 & 1 \\ \vdots & \\ I_{|w_k|} & 1 \\ \sqrt{\epsilon} & 0 \end{pmatrix}\left\{\begin{pmatrix} I_1 & I_2 & \ldots & I_{|w_k|} & \sqrt{\epsilon} \\ 1 & 1 & & 1 & 0 \end{pmatrix}\begin{pmatrix} I_1 & 1 \\ I_2 & 1 \\ \vdots & \\ I_{|w_k|} & 1 \\ \sqrt{\epsilon} & 0 \end{pmatrix}\right\}^{-1}\begin{pmatrix} I_1 & I_2 & \ldots & I_{|w_k|} & \sqrt{\epsilon} \\ 1 & 1 & & 1 & 0 \end{pmatrix}$$

Compute the Inverse

$$\begin{pmatrix} \sum_n^{|w_k|} I_n^2 + \epsilon & \sum_n^{|w_k|} I_n \\ \sum_n^{|w_k|} I_n & |w_k| \end{pmatrix}^{-1}$$

$$= \frac{\begin{pmatrix} |w_k| & -\sum_n^{|w_k|} I_n \\ -\sum_n^{|w_k|} I_n & \sum_n^{|w_k|} I_n^2 + \epsilon \end{pmatrix}}{|w_k| \sum_n^{|w_k|} I_n^2 + \epsilon |w_k| - (\sum_n^{|w_k|} I_n)^2}$$

$$= \frac{|w_k| \begin{pmatrix} 1 & -\mu_k \\ -\mu_k & \sum_n^{|w_k|} I_n^2 / |w_k| + \epsilon / |w_k| \end{pmatrix}}{|w_k|^2 \sigma_k^2 + \epsilon |w_k|}$$

$$= \frac{1}{|w_k| \sigma_k^2 + \epsilon} \begin{pmatrix} 1 & -\mu_k \\ -\mu_k & \sum_l^{|w_k|} I_n^2 / |w_k| + \epsilon / |w_k| \end{pmatrix}$$

The (i,j) element

$$\left(\mathbf{I} - G_k \left(G_k^T G_k\right)^{-1} G_k^T\right)_{ij}$$

$$= \delta_{ij} - (I_i\ 1) \begin{pmatrix} \sum_n^{|w_k|} I_n^2 + \epsilon & \sum_n^{|w_k|} I_n \\ \sum_n^{|w_k|} I_n & |w_k| \end{pmatrix}^{-1} \begin{pmatrix} I_j \\ 1 \end{pmatrix}$$

$$= \delta_{ij} - (I_i\ 1) \frac{1}{|w_k|\sigma_k^2 + \epsilon} \begin{pmatrix} 1 & -\mu_k \\ -\mu_k & \sum_n^{|w_k|} I_n^2/|w_k| + \epsilon/|w_k| \end{pmatrix} \begin{pmatrix} I_j \\ 1 \end{pmatrix}$$

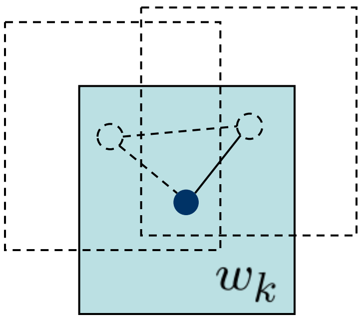
$$= \delta_{ij} - \frac{1}{|w_k|\sigma_k^2 + \epsilon} \left(I_i I_j - I_i \mu_k - I_j \mu_k + \frac{\sum_n^{|w_k|} I_n^2 + \epsilon}{|w_k|} \right)$$

$$= \delta_{ij} - \frac{1}{|w_k|\sigma_k^2 + \epsilon} \left(I_i I_j - I_i \mu_k - I_j \mu_k + \mu_k^2 + \frac{\sum_n^{|w_k|} I_n^2}{|w_k|} - \mu_k^2 + \frac{\epsilon}{|w_k|} \right)$$

$$= \delta_{ij} - \frac{1}{|w_k|\sigma_k^2 + \epsilon} \left((I_i - \mu_k)(I_j - \mu_k) + \sigma_k^2 + \frac{\epsilon}{|w_k|} \right)$$

$$= \delta_{ij} - \frac{1}{|w_k|} \left(1 + \frac{1}{\sigma_k^2 + \epsilon/|w_k|} (I_i - \mu_k)(I_j - \mu_k) \right)$$

$$J(\alpha) = \sum_k \bar{\alpha}_k^T \bar{G}_k^T \bar{G}_k \bar{\alpha}_k$$



$$J(\alpha) = \alpha^T L \, \alpha$$

L is a large sparse N-by-N matrix whose (i, j) element is

$$\sum_{k|(i,j)\in w_k} \left(\delta_{ij} - \frac{1}{|w_k|} \left(1 + \frac{1}{\sigma_k^2 + \epsilon/|w_k|} (I_i - \mu_k)(I_j - \mu_k) \right) \right)$$

N is the number of pixels in the image

Color Images

color line model

$$\alpha_i \approx \sum_c a^c I_i^c \; + b$$

sum over color channels

$$F_i = \beta_i^F F_1 + (1 - \beta_i^F) F_2$$

linear mixture of two colors

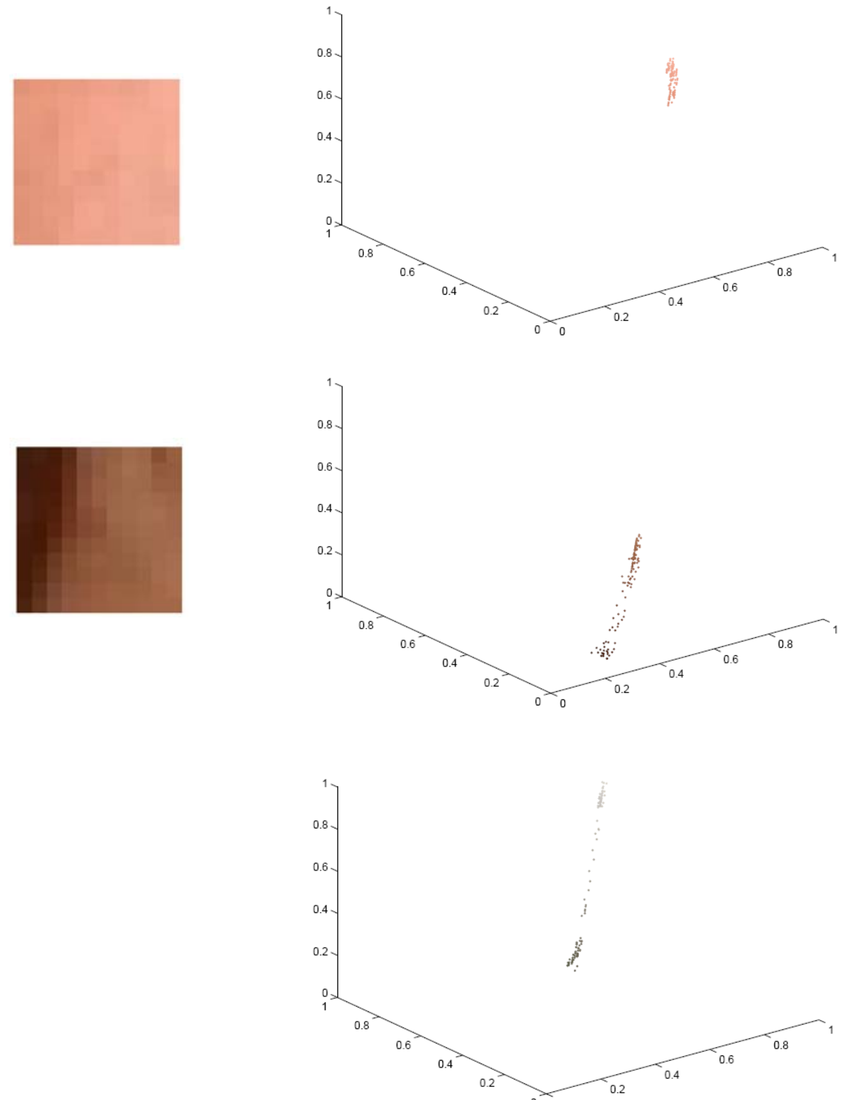
$$B_i = \beta_i^B B_1 + (1 - \beta_i^B) B_2$$

$$I_i^c = \alpha_i(\beta_i^F F_1^c + (1 - \beta_i^F) F_2^c) + (1 - \alpha_i)(\beta_i^B B_1^c + (1 - \beta_i^B) B_2^c)$$

$$H \begin{bmatrix} \alpha_i \\ \alpha_i \beta_i^F \\ (1 - \alpha_i)\beta_i^B \end{bmatrix} = I_i - B_2$$

$$J(\alpha) = \sum_k \bar{\alpha}_k^T \bar{G}_k^T \bar{G}_k \bar{\alpha}_k$$

Color Line Model



Alpha Matting of Color Images

$$J(\alpha, a, b) = \sum_{j \in I} \left(\sum_{i \in w_j} \left(\alpha_i - \sum_c a_j^c I_i^c - b_j \right)^2 + \epsilon \sum_c {a_j^c}^2 \right)$$

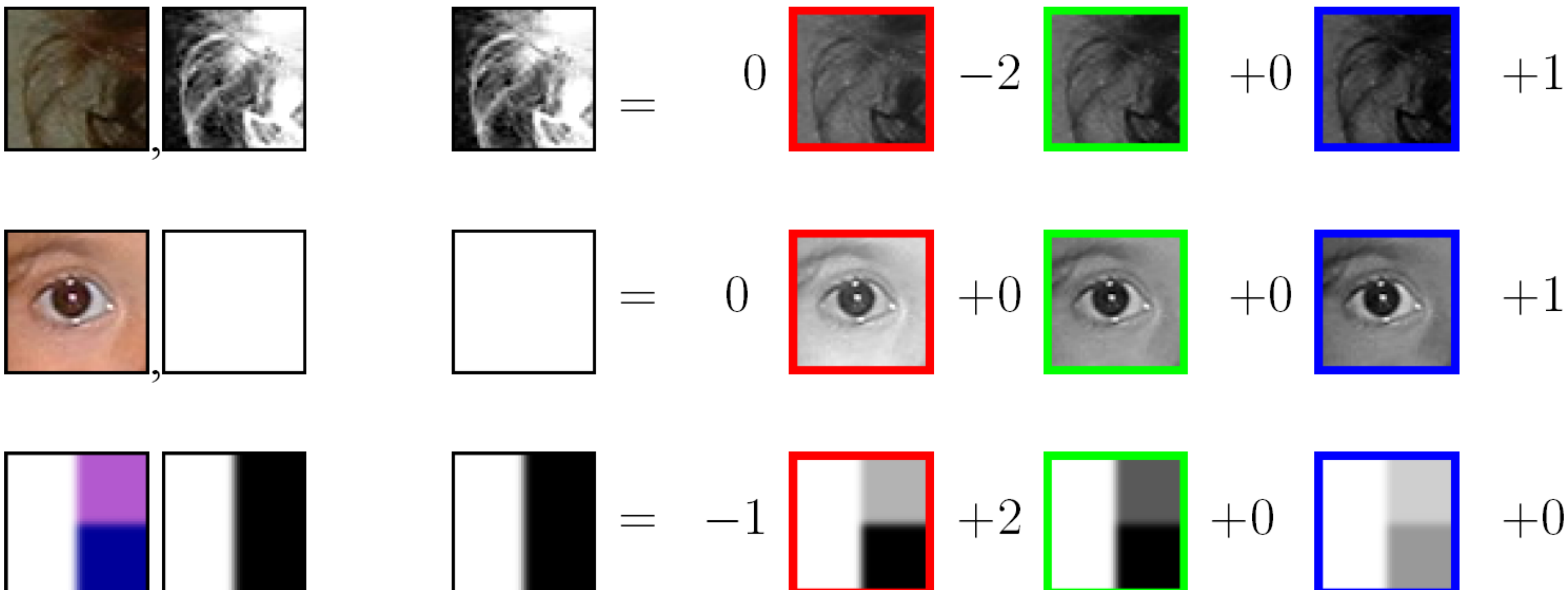
$$J(\alpha) = \alpha^T L \, \alpha$$

L is a large sparse N-by-N matrix whose (i, j) element is

$$\sum_{k | (i,j) \in w_k} \left(\delta_{ij} - \frac{1}{|w_k|} \left(1 + (I_i - \mu_k)^T (\Sigma_k + \frac{\epsilon \mathbf{I}_3}{|w_k|})^{-1} (I_j - \mu_k) \right) \right)$$

where N is the number of pixels in the image, Σ_k is a 3-by-3 covariance matrix and μ_k a 3-by-1 mean vector of the colors in a window w_k, and $\mathbf{I}_3$ is the 3-by-3 identity matrix.

$$\alpha_i \approx \sum_c a^c I_i^c \; + b$$



Constraints and User Interface

$$\alpha = \operatorname{argmin}\ \alpha^T L\, \alpha + \lambda(\alpha^T - b_S^T) D_S\, (\alpha - b_S)$$

vector of scribbles
0: background
1: foreground

diagonal matrix
1: constrained pixels

solving the sparse linear system

$$(L + \lambda D_S)\alpha = \lambda b_S$$

Reconstructing F and B

$$min \sum_{i \in I} \sum_{c} \left(\alpha_i F_i^c + (1-\alpha_i) B_i^c - I_i^c\right)^2$$

$$+ \quad |\alpha_{i_x}| \left((F_{i_x}^c)^2 + (B_{i_x}^c)^2\right) + |\alpha_{i_y}| \left((F_{i_y}^c)^2 + (B_{i_y}^c)^2\right)$$

x derivates

- Introducing some smoothness priors
- The smoothness priors are stronger in the presence of matte edges